

POLYNOMIAL INVARIANTS OF TORUS KNOTS AND (p, q) -CALCULUS¹

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Abstract

We introduce the deformed fermionic numbers, corresponding to the skein relations, the main characteristics of knots and links. These fermionic numbers allow one to restore the skein relations. For the Alexander (Jones) skein relation we introduce corresponding Alexander (Jones) fermionic q -numbers, and for the HOMFLY skein relation – the HOMFLY deformed (p, q) -numbers with one fermionic parameter.

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1 Introduction

The problem of searching, investigating and physical interpreting of the skein relations for knots and links is rather difficult one. Because of this we propose to deal with the corresponding deformed (bosonic, fermionic) numbers, instead of working with the skein relations.

In the recent paper [1] we have shown that the one-parameter Alexander (Jones) skein relation is characterized by the corresponding Alexander (Jones) bosonic q -numbers. The two-parameter HOMFLY skein relation is described by the two-parameter HOMFLY bosonic (p, q) -numbers. The introduced bosonic q -numbers/ (p, q) -numbers allow one to restore corresponding skein relations.

In the present paper we show that the Alexander (Jones) skein relation can be defined by the corresponding Alexander (Jones) fermionic q -numbers as well. The HOMFLY skein relation is described by the two-parameter HOMFLY deformed (p, q) -numbers with one fermionic parameter. The introduced deformed fermionic numbers allow us to restore the skein relations as well.

2 Skein Relation

Skein relation, the most important characteristics of knots and links, unites the three polynomials $P_{L_+}(q)$, $P_{L_O}(q)$, $P_{L_-}(q)$, and in the general form can be written as

$$P_{L_+}(q) = l_1 P_{L_O}(q) + l_2 P_{L_-}(q), \quad (1)$$

where l_1, l_2 are coefficients. The capital letter "L" stands for "Link", i.e. knot or link. The L_+, L_O, L_- denote Link with overcrossing, Link with zero crossing and Link with undercrossing correspondingly. The initial Link L_+ turns into a simpler Link L_O by the surgery operation of elimination applied to chosen overcrossing of L_+ . The same initial Link L_+ turns into another simpler Link L_- by the surgery operation of switching applied to the same chosen overcrossing of L_+ .

Applying the surgery operations to $L_{n+1,2}$, the simplest torus knots (if $(n+1)$ is odd) and links (if $(n+1)$ is even integer number), one obtains

$L_{n,2}$ (by elimination) and $L_{n-1,2}$ (by switching). Expressing it in terms of (1), we have the recurrence relation [2]

$$P_{L_{n+1,2}}(q) = l_1 P_{L_{n,2}}(q) + l_2 P_{L_{n-1,2}}(q), \quad (2)$$

which has the same coefficients as the skein relation (1), or in simpler form

$$P_{n+1,2}(q) = l_1 P_{n,2}(q) + l_2 P_{n-1,2}(q). \quad (3)$$

The axiomatic basis of the knot theory includes the skein relation (1), and the normalization condition

$$P_{unknot} = P_{1,2} = 1. \quad (4)$$

3 Two-parameter deformed numbers

The two-parameter (P, Q) -number for an integer n appearing in connection with two-parameter deformed oscillator is defined as [3]

$$[n]_{P,Q} = \frac{P^n - Q^n}{P - Q}. \quad (5)$$

Some of the first (P, Q) -numbers are given here:

$$[1]_{P,Q}=1, \quad [2]_{P,Q}=P+Q, \quad [3]_{P,Q}=P^2+PQ+Q^2, \quad [4]_{P,Q}=P^3+P^2Q+PQ^2+Q^3, \dots$$

The recurrence relation for (P, Q) -numbers looks as

$$[n+1]_{P,Q} = (P+Q)[n]_{P,Q} - PQ[n-1]_{P,Q}. \quad (6)$$

Comparing (3) (or directly (1)) with (6) allows one to introduce q -numbers, corresponding to the skein relation (1). For this it is necessary to find P and Q from

$$P+Q = l_1, \quad PQ = -l_2, \quad (7)$$

and put them into (5).

4 Alexander q -numbers

The Alexander skein relation [4]

$$\Delta_+(q) - \Delta_-(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\Delta_O(q) \quad (8)$$

defines the Alexander polynomials $\Delta(q)$ for knots and links. Rewriting it in the form (1)

$$\Delta_+(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\Delta_O(q) + \Delta_-(q) \quad (9)$$

gives the "Link coefficients"

$$l_1^A = q^{\frac{1}{2}} - q^{-\frac{1}{2}}, \quad l_2^A = 1. \quad (10)$$

In analogy to (3), we have the recurrence relation for Alexander polynomials of torus knots and links $L_{n,2}$:

$$\Delta_{n+1,2}(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\Delta_{n,2}(q) + \Delta_{n-1,2}(q). \quad (11)$$

Comparing (11) and (6) one has two equations

$$P + Q = q^{\frac{1}{2}} - q^{-\frac{1}{2}}, \quad PQ = -1, \quad (12)$$

from which it follows

$$P = q^{\frac{1}{2}}, \quad Q = -q^{-\frac{1}{2}}. \quad (13)$$

Putting (13) into (5), we obtain the Alexander deformed fermionic q -numbers

$$[n]_{q^{\frac{1}{2}}, -q^{-\frac{1}{2}}}^A = \frac{(q^{\frac{1}{2}})^n - (-q^{-\frac{1}{2}})^n}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}. \quad (14)$$

The Alexander q -numbers $[n]_{q^{\frac{1}{2}}, -q^{-\frac{1}{2}}}^A$, which are described by the below indices, satisfy the following recurrence relation

$$[n+1]_{q^{\frac{1}{2}}, -q^{-\frac{1}{2}}}^A = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})[n]_{q^{\frac{1}{2}}, -q^{-\frac{1}{2}}}^A + [n-1]_{q^{\frac{1}{2}}, -q^{-\frac{1}{2}}}^A,$$

from which we find the Alexander "Link coefficients" (10), and by putting them into (3) and (1) we obtain (11) and (9).

The Alexander polynomial invariants for torus knots $T(n, l)$ are given by the known formula [5]

$$\Delta_{n,l}(q) = \frac{(q^{\frac{nl}{2}} - q^{-\frac{nl}{2}})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(q^{\frac{l}{2}} - q^{-\frac{l}{2}})}, \quad (15)$$

where n and l are coprime positive integers. For $l = 2$, Eq. (15) gives for torus knots $T(n, 2)$:

$$\Delta_{n,2}(q) = \frac{q^{\frac{n}{2}} + q^{-\frac{n}{2}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}, \quad n = 2m - 1, \quad (16)$$

where m are positive integers: $m = 1, 2, 3, \dots$. Thus, from (16) and (14) for torus knots $T(n, 2)$:

$$\Delta_{n,2}(q) = [n]_{q^{\frac{1}{2}}, -q^{-\frac{1}{2}}}^A, \quad n = 2m - 1. \quad (17)$$

It is easy to verify that (14) in the case of even n gives the formula of the Alexander polynomials for torus links $L(n, 2)$:

$$\Delta_{n,2}(q) = [n]_{q^{\frac{1}{2}}, -q^{-\frac{1}{2}}}^A = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}, \quad n = 2m. \quad (18)$$

Thus, uniting (16) and (18), one has the Alexander polynomials for torus knots and links $L_{n,2}$:

$$\Delta_{n,2}(q) = [n]_{q^{\frac{1}{2}}, -q^{-\frac{1}{2}}}^A, \quad n = 1, 2, 3, \dots \quad (19)$$

So, we introduced the Alexander fermionic q -numbers (14), allowing to find the Alexander skein relation (9). At last, we recall that the Alexander bosonic q -numbers, which also lead to the Alexander skein relation, were introduced as well [1]:

$$[n]_{q, q^{-1}}^A = \frac{q^n - q^{-n}}{q - q^{-1}}. \quad (20)$$

They coincide with q -numbers of Biedenharn and Macfarlane [6, 7]. From the recurrence relation for the Alexander bosonic q -numbers (20)

$$[n + 1]_{q, q^{-1}}^A = (q + q^{-1})[n]_{q, q^{-1}}^A - [n - 1]_{q, q^{-1}}^A$$

one has the "knot coefficients":

$$k_1^A = q + q^{-1}, \quad k_2^A = -1,$$

from which the "Link coefficients" (10), defining the Alexander skein relation (9), can be found with the help of the formulas

$$l_2^A = +(-k_2^A)^{\frac{1}{2}}, \quad l_1^A = +(k_1^A - 2l_2^A)^{\frac{1}{2}}.$$

5 Jones q -numbers

The Jones skein relation [8]

$$q^{-1}V_+(q) - qV_-(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})V_O(q) \quad (21)$$

introduces the Jones polynomials $V(q)$ for knots and links. From (21) in the form (1)

$$V_+(q) = q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})V_O(q) + q^2V_-(q) \quad (22)$$

one finds the Jones "Link coefficients"

$$l_1^V = q^{\frac{3}{2}} - q^{\frac{1}{2}}, \quad l_2^V = q^2. \quad (23)$$

Comparing (22) and (6) gives

$$P + Q = q^{\frac{3}{2}} - q^{\frac{1}{2}}, \quad PQ = -q^2, \quad (24)$$

from which

$$P = q^{\frac{3}{2}}, \quad Q = -q^{\frac{1}{2}}. \quad (25)$$

Putting (25) into (5), we obtain the Jones fermionic q -numbers

$$[n]_{q^{\frac{3}{2}}, -q^{\frac{1}{2}}}^V = \frac{(q^{\frac{3}{2}})^n - (-q^{\frac{1}{2}})^n}{q^{\frac{3}{2}} + q^{\frac{1}{2}}}, \quad (26)$$

which satisfy the following recurrence relation

$$[n+1]_{q^{\frac{3}{2}}, -q^{\frac{1}{2}}}^V = (q^{\frac{3}{2}} - q^{\frac{1}{2}})[n]_{q^{\frac{3}{2}}, -q^{\frac{1}{2}}}^V + q^2[n-1]_{q^{\frac{3}{2}}, -q^{\frac{1}{2}}}^V,$$

defining the Jones "Link coefficients" (23), and, therefore, the Jones skein relation (22) follows from (1).

Finally, we recall that the Jones bosonic q -numbers [9, 1] are

$$[n]_{q^3, q}^V = \frac{q^{3n} - q^n}{q^3 - q}. \quad (27)$$

The recurrence relation for the Jones bosonic q -numbers (27)

$$[n+1]_{q^3, q}^V = (q^3 + q)[n]_{q^3, q}^V - q^4[n-1]_{q^3, q}^V$$

gives the "knot coefficients":

$$k_1^V = q^3 + q, \quad k_2^V = q^4,$$

from which the "Link coefficients" (23), leading to (22), follow

$$l_2^V = +(-k_2^V)^{\frac{1}{2}}, \quad l_1^V = +(k_1^V - 2l_2^V)^{\frac{1}{2}}.$$

6 HOMFLY (p, q) -numbers

The HOMFLY skein relation [10]

$$p^{-1}H_+(p, z) - pH_-(p, z) = zH_O(p, z) \quad (28)$$

defines the two-parameter HOMFLY polynomials $H(p, z)$. Let us make a change of the variable

$$z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}.$$

Rewriting (28) in the form (1),

$$H_+(p, q) = p(q^{\frac{1}{2}} - q^{-\frac{1}{2}})H_O(p, q) + p^2H_-(p, q), \quad (29)$$

one obtains the "Link coefficients"

$$l_1^H = p(q^{\frac{1}{2}} - q^{-\frac{1}{2}}), \quad l_2^H = p^2. \quad (30)$$

Then from

$$P + Q = p(q^{\frac{1}{2}} - q^{-\frac{1}{2}}), \quad PQ = -p^2, \quad (31)$$

one has

$$P = pq^{\frac{1}{2}}, \quad Q = -pq^{-\frac{1}{2}}. \quad (32)$$

Thus, we obtain the HOMFLY fermionic q -numbers

$$[n]_{pq^{\frac{1}{2}}, -pq^{-\frac{1}{2}}}^H = p^{n-1} \cdot \frac{(q^{\frac{1}{2}})^n - (-q^{-\frac{1}{2}})^n}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} = p^{n-1} \cdot [n]_{q^{\frac{1}{2}}, -q^{-\frac{1}{2}}}^A. \quad (33)$$

And, at the end, we remind that the HOMFLY bosonic q -numbers look as

$$[n]_{p^2q, p^2q^{-1}}^H = p^{2(n-1)} \cdot \frac{q^n - q^{-n}}{q - q^{-1}}. \quad (34)$$

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